N-point matrix elements of dynamical vertex operators of the higher spin XXZ model

A H Bougourzi†

Centre de Recherches Mathématiqes, Université de Montréal, CP 6128-A, Montréal, Québec H3C 3J7, Canada

Abstract. We extend the concept of conjugate vertex operators, first introduced by Dotsenko in the case of the bosonization of the SU(2) conformal field theory, to the bosonization of the dynamical vertex operators (type II in the classification of the Kyoto school) of the higher spin XXZ model. We show that the introduction of the conjugate vertex operators leads to simpler expressions for the N-point matrix elements of the dynamical vertex operators, that is, without redundant Jackson integrals that arise from the insertion of screening charges. In particular, the two-point matrix element can be represented without any integral.

1. Introduction

It is now well established through the work of the Kyoto school that the XXZ quantum spin chain with local spins equal to $\frac{1}{2}k$ has the quantum affine algebra $U_q(\widehat{su}(2))$ with level k as a non-Abelian symmetry; q being a deformation (anisotropy) parameter [1–3]. The dynamical symmetry of this model is generated by the dynamical vertex operators (referred to as type II vertex operators in [1–3]) which create non-degenerate eigenstates of the XXZ Hamiltonian by a successive action on a given eigenvector, for example, the vacuum. A dynamical vertex operator has two main properties: it intertwines the $U_q(\widehat{su}(2))$ modules and carries spin $\frac{1}{2}$ [3].

Important mathematical and physical quantities in this model is the N-point matrix elements of the above vertex operators. There are three known ways to compute these matrix elements exactly, at least in principle. The first one is by solving the q-KZ equation. Indeed, it has been shown in [4] that these vertex operators satisfy a difference equation, which is the q-analogue of the usual KZ equation in conformal field theory. The second one consists of deriving a normal ordering of the modes of the vertex operators that is compatible with the Zamolodchikov–Fateev algebra they satisfy. With this normal ordering one can then in principle compute any matrix element. The third one consists in realizing the vertex operators in terms of bosonic modes satisfying Heisenberg algebras. Since the normal ordering of the latter modes is very simple, one can use it to compute matrix elements.

In practice however, the first two methods are hardly useful beyond the two-point matrix elements because of highly technical complications, whereas the third one, though more useful and systematic, also raises the following problem: a single vertex operators might have several independent realizations in terms of bosonic modes. Equivalently, a given $U_q(\widehat{su}(2))$ representation might be identified with several Fock spaces with different bosonic

† E-mail: bougourz@ere.umontreal.ca

charges (eigenvalues with respect to the centre of the Heisenberg algebra). Therefore, it is not trivial which combinations of these realizations of the vertex operators and Fock spaces are going to lead to the right matrix elements. This problem has been addressed in the context of the bosonization of conformal field theory through two different approaches [5], which we extend to the case of $U_q(\widehat{su}(2))$ algebra. The first one consists of singling out only one particular realization, say the simplest, for the vertex operators and then attaching an appropriate number of screening charges to these vertex operators so that the resulting operator is a map connecting to the two Fock spaces between which we aim to compute the matrix elements. This amounts to fixing the same bosonic charge (picture fixing) for both Fock spaces realizing a $U_q(\widehat{su}(2))$ representation and its dual. The main disadvantage of this method is that the final expressions for the matrix elements involve in general redundant integrations coming from the screening charges. The second method, which is due to Dotsenko (in the classical case i.e., $q = 1$) [5], consists of deriving all possible independent bosonizations of the vertex operators. Then one should single out two among them such that the two-point matrix element can be computed as the expectation value of the product of the two (i.e., each one of them is inserted once) and without the insertion of screening charges. This is equivalent to fixing two different bosonic charges: one for the Fock space realizing a $U_q(\widehat{su}(2))$ representation and the other for the Fock space realizing its dual. This is the reason why Dotsenko refers to one of them as a vertex operator and the other one as its conjugate vertex operator, i.e., they create respectively a representation and its dual space. From this bosonic realization of the two-point matrix element one reads off the conservation law of the bosonic charges. This conservation law must be obviously satisfied in the N-point matrix elements otherwise one must again insert the minimum number of screening charges in the matrix elements to make it so. This second method has the main advantage of avoiding unnecessary redundant integrations in the integral representations of the matrix elements obtained through bosonization.

So far only the first method has been applied in the computation of matrix elements of the dynamical vertex operators of the spin $\frac{1}{2}k$ XXZ model [6]. In this paper, we apply the second method to the computation of these N-point matrix elements. In section 2, we briefly review the $U_q(\widehat{su}(2))$ algebra and its bosonic realization. In section 3, we recall the definition of the dynamical vertex operators as intertwiners of $U_q(\widehat{su}(2))$ modules. The bosonization of these vertex operators are derived by solving the intertwining relations they satisfy with the $U_q(\widehat{su}(2))$ algebra, which is already in a bosonized form. We show that here also there are two independent solutions, just as in conformal field theory [5]. One of them has already been derived in [6], whereas the second one is new. We refer to the first solution as the vertex operator and to the second as the conjugate vertex operator. In section 4, we show how the N-point vacuum-to-vacuum matrix element can be computed without a redundant integration if both the vertex operators and conjugate vertex operators are used simultaneously. As an explicit example, we compute the two-point matrix element in a non-integral form and show that it satisfies the q-KZ equation. Finally section 5 is devoted to our conclusions.

2. The $U_q(\widehat{su}(2))$ algebra and its bosonization

The associative initial $U_q(\widehat{su}(2))$ algebra is generated by the elements $\{E_n^{\pm}(n \in Z), H_m(m \in Z \{0\}, K^{\pm 1}, \gamma^{\pm 1/2}\}$, with the following defining relations [7]:

$$KK^{-1} = K^{-1}K = 1$$
$$\gamma^{\frac{1}{2}}\gamma^{-\frac{1}{2}} = \gamma^{-\frac{1}{2}}\gamma^{\frac{1}{2}} = 1$$
$$[K^{\pm 1}, H_m] = 0$$
$$KE_n^{\pm}K^{-1} = q^{\pm 2}E_n^{\pm}$$
$$[H_n, H_m] = \frac{[2n]}{2n}\frac{\gamma^n - \gamma^{-n}}{q - q^{-1}}\delta_{n+m,0} \tag{2.1}$$
$$[H_n, E_m^{\pm}] = \pm\sqrt{2}\frac{\gamma^{\mp|n|/2}[2n]}{2n}E_{n+m}^{\pm}$$
$$[E_n^+, E_m^-] = \frac{\gamma^{(n-m)/2}\psi_{n+m} - \gamma^{(m-n)/2}\varphi_{n+m}}{q - q^{-1}}$$
$$E_{n+1}^{\pm}E_m^{\pm} - q^{\pm 2}E_m^{\pm}E_{n+1}^{\pm} = q^{\pm 2}E_n^{\pm}E_{m+1}^{\pm} - E_{m+1}^{\pm}E_n^{\pm}$$

where $\gamma^{1/2}$ is in the centre of the algebra and acts as $q^{k/2}$ on level k highest weight representations of $U_q(\widehat{su}(2))$, and ψ_n and φ_n are the modes of the fields $\psi(z)$ and $\varphi(z)$ defined by

$$\psi(z) = \sum_{n\geqslant 0}\psi_n z^{-n} = q^{\sqrt{2}H_0}\exp\left\{\sqrt{2}(q - q^{-1})\sum_{n>0}H_n z^{-n}\right\}$$
$$\varphi(z) = \sum_{n\leqslant 0}\varphi_n z^{-n} = q^{-\sqrt{2}H_0}\exp\left\{-\sqrt{2}(q - q^{-1})\sum_{n<0}H_n z^{-n}\right\}. \tag{2.2}$$

As usual, $[x]$ is defined by $[x] = (q^x - q^{-x})/(q - q^{-1})$ and q is the deformation parameter. The above algebra is a Hopf algebra with the following comultiplication:

$$\Delta(E_n^+) = E_n^+ \otimes \gamma^n + \gamma^{2n}K \otimes E_n^+ + \sum_{i=0}^{n-1}\gamma^{(n+3i)/2}\psi_{n-i} \otimes \gamma^{n-i}E_i^+ \bmod N_- \otimes N_+^2$$

$$\Delta(E_{-m}^+) = E_{-m}^+ \otimes \gamma^{-m} + K^{-1} \otimes E_{-m}^+ + \sum_{i=0}^{m-1}\gamma^{(m-i)/2}\varphi_{-m+i} \otimes \gamma^{i-m}E_{-i}^+ \bmod N_- \otimes N_+^2$$

$$\Delta(E_{-n}^-) = E_{-n}^- \otimes \gamma^{-2n}K^{-1} + \gamma^{-n} \otimes E_{-n}^- + \sum_{i=0}^{n-1}\gamma^{i-n}E_i^- \otimes \gamma^{-(n+3i)/2}\varphi_{i-n} \bmod N_-^2 \otimes N_+$$

$$\Delta(E_m^-) = \gamma^m \otimes E_m^- + E_m^- \otimes K + \sum_{i=1}^{m-1}\gamma^{m-1}E_m^- \otimes \gamma^{(i-m)/2}\psi_{m-i} \bmod N_-^2 \otimes N_+$$

$$\Delta(H_m) = H_m \otimes \gamma^{m/2} + \gamma^{3m/2} \otimes H_m \bmod N_- \otimes N_+$$
$$\Delta(H_{-m}) = H_{-m} \otimes \gamma^{-3m/2} + \gamma^{-m/2} \otimes H_{-m} \bmod N_- \otimes N_+$$
$$\Delta(K^{\pm 1}) = K^{\pm 1} \otimes K^{\pm 1}$$
$$\Delta(\gamma^{\pm 1/2}) = \gamma^{\pm 1/2} \otimes \gamma^{\pm 1/2} \tag{2.3}$$

where $m > 0$, $n \geqslant 0$, and $N_{\pm}$ and $N_{\pm}^2$ are left $Q[\gamma^{\pm}, \psi_m, \varphi_{-n}; m, n \in \mathbb{Z}_{\geqslant 0}]$-modules generated by $\{E_m^{\pm}; m \in \mathbb{Z}\}$ and $\{E_m^{\pm}E_n^{\pm}; m, n \in \mathbb{Z}\}$ respectively [8]. This comultiplication will be useful in deriving the intertwining properties of the vertex operators.

Let us now briefly review the bosonization of $U_q(\widehat{su}(2))$ for arbitrary level k. We need three deformed Heisenberg algebras generated by the elements $\{a_n^j, j = 1, 2, 3; n \in \mathbb{Z}\}$, to

which we adjoin the elements a^j, $j = 1, 2, 3$, and with the following defining relations [9]:

$$[a_n^j, a_m^\ell] = (-1)^{j-1} n I_j(n) \delta^{j,\ell} \delta_{n+m,0}$$
$$[a^j, a_0^\ell] = (-1)^{j-1} \mathrm{i} \delta^{j,\ell} \qquad j, \ell = 1, 2, 3 \qquad (2.4)$$

and where

$$I_1(n) = \frac{[2n][nk]}{2kn^2} \qquad I_2(n) = \frac{[nk][n(2+k)]}{n^2 k(2+k)} q^{|n|k} \qquad I_3(n) = \frac{[2n]^2}{4n^2}. \qquad (2.5)$$

The bosonization of $U_q(\widehat{su}(2))$ is given by [9]

$$\psi(z) = \exp\left\{ \mathrm{i}\sqrt{\frac{2}{k}} (\chi^{1,+}(zq^{k/2}) - \chi^{1,-}(zq^{-k/2})) \right\}$$

$$\varphi(z) = \exp\left\{ \mathrm{i}\sqrt{\frac{2}{k}} (\chi^{1,+}(zq^{-k/2}) - \chi^{1,-}(zq^{k/2})) \right\}$$

$$E^+(z) = \frac{1}{z(q - q^{-1})} (E_+^+(z) - E_-^+(z))$$

$$E_\pm^+(z) = \exp\left\{ \mathrm{i}\sqrt{\frac{2}{k}} \chi^{1,+}(z) + \mathrm{i}\sqrt{\frac{2+k}{k}} \chi^2(z) - \mathrm{i}\chi^3(zq^{\mp 1}) \right\} \qquad (2.6)$$

$$E^-(z) = \frac{1}{z(q - q^{-1})} (E_+^-(z) - E_-^-(z))$$

$$E_\pm^-(z) = \exp\left\{ -\mathrm{i}\sqrt{\frac{2}{k}} \chi^{1,-}(z) - \mathrm{i}\sqrt{\frac{2+k}{k}} \chi^2(zq^{\pm k}) + \mathrm{i}\chi^3(zq^{\pm(1+k)}) \right\}$$

where z is a complex variable and

$$E^\pm(z) = \sum_{n \in Z} E_n^\pm z^{-n-1}$$

$$\chi^{1,\pm}(z) = \Omega^1(k, 1; 1|0, \mp \tfrac{1}{2}k/2, 0| + |z) \qquad (2.7)$$

$$\chi^2(z) = \Omega^2(k, 1; 1|0, -\tfrac{1}{2}k, 0| + |z)$$

$$\chi^3(z) = \Omega^3(2, 1; 1|0, 0, k + 2| + |z).$$

Here, we have used the notation

$$\Omega^j(L_1, L_2, \ldots, L_r; M_1, M_2, \ldots, M_s, |\sigma, \alpha, \beta| \pm |z)$$
$$= a^j - \mathrm{i}a_0^j \ln(\pm zq^\sigma) + \mathrm{i} \frac{L_1 L_2 \ldots L_r}{M_1 M_2 \ldots M_s}$$
$$\times \sum_{n \in Z \backslash \{0\}} \frac{[M_1 n][M_2 n] \ldots [M_s n]}{[L_1 n][L_2 n] \ldots [L_r n]} a_n^j q^{\alpha|n| + \beta n} z^{-n}. \qquad (2.8)$$

where $L_1, L_2, \ldots L_r$, $M_1, M_2, \ldots, M_s$, σ, α and β are parameters related to the q-deformation.

3. Bosonization of the dynamical vertex operators

Here we consider the bosonization of the dynamical vertex operators and introduce the concept of their conjugate vertex operators. These vertex operators are referred to as type II vertex operators in [3]. They map the $U_q(\widehat{su}(2))$ modules in the following way:

$$\Phi_{j_1}^{j\, j_2}(z) : V(\Lambda_{j_1}) \to V^j(z) \otimes V(\Lambda_{j_2}) \qquad (3.9)$$

where $V(\Lambda_j)$ are level k highest weight $U_q(\widehat{su}(2))$-modules and $\{\Lambda_j = (k-2j)\lambda_0 + 2j\lambda_1, j = 0, \ldots, k/2\}$ and $\{\lambda_0, \lambda_1\}$ denote the sets of $U_q(\widehat{su}(2))$ dominant highest weights and fundamental weights respectively. $V^J(z)$ is the $k = 0$ 'evaluation representation' of $U_q(\widehat{su}(2))$. It is isomorphic to $V^j \otimes C[z, z^{-1}]$, where V^j is the $2j+1$ dimensional representation with the basis $\{v^j_m, -j \leqslant m \leqslant j\}$, and is equipped with the following $U_q(\widehat{su}(2))$-module structure:

$$\gamma^{\pm 1/2} v^j_m \otimes z^\ell = v^j_m \otimes z^\ell$$

$$q^{\sqrt{2}H_0} v^J_m \otimes z^\ell = q^{-2m} v^j_m \otimes z^\ell$$

$$E^+_n v^j_m \otimes z^\ell = q^{2n(1-m)}[j+m] v^j_{m-1} \otimes z^{\ell+n} \tag{3.10}$$

$$E^-_n v^j_m \otimes z^\ell = q^{-2nm}[j-m] v^j_{m+1} \otimes z^{\ell+n}$$

$$H_n v^j_m \otimes z^\ell = \frac{1}{\sqrt{2}n} \{[2nj] - q^{n(j-m+1)}(q^n + q^{-n})[n(j+m)]\} v^j_m \otimes z^{\ell+n}$$

with v^j_m being identically zero if $|m| > j$.

Let us introduce the rescaled vertex operators $\tilde{\Phi}^{j\,j_2}_{j_1}(z)$ as

$$\Phi^{j\,j_2}_{j_1}(z) = z^{(\Delta_{j_2} - \Delta_{j_1})} \tilde{\Phi}^{j\,j_2}_{j_1}(z) \tag{3.11}$$

where $\Delta_j = j(j+1)/(k+2)$. The latter vertex operators are defined to obey the following intertwining relations [4, 3]:

$$\tilde{\Phi}^{j\,j_2}_{j_1}(z) \circ x = \Delta(x) \circ \tilde{\Phi}^{j\,j_2}_{j_1}(z) \qquad \forall x \in U_q(\widehat{su}(2)) \tag{3.12}$$

where Δ is the comultiplication given in (2.3). We define the components $\phi^j_m(z)$ of these vertex operators as

$$\tilde{\Phi}^{j\,j_2}_{j_1}(z) = g^{j\,j_2}_{j_1}(z) \sum_{m=-j}^{j} v^j_m \otimes \phi^j_m(z) \tag{3.13}$$

where the normalization function $g^{j\,j_2}_{j_1}(z)$ is to be determined so that

$$\tilde{\Phi}^{j\,j_2}_{j_1}(z)|\Lambda_{j_2}\rangle = |\Lambda_{j_2}\rangle + \cdots \tag{3.14}$$

with $|\Lambda_{j_1}\rangle$ and $|\Lambda_{j_2}\rangle$ being the highest weight states of $V(\Lambda_{j_1})$ and $V(\Lambda_{j_2})$ respectively.

Using relation (3.12), the comultiplication (2.3), and the fact that $N_+ v^j_{-j} = N_- v^j_j = 0$, $N_\pm v^j_m \in C[z, z^{-1}] v^j_{m \mp 1}$, we get the following commutation relations:

$$[E^-_n, \phi^j_{-j}(z)] = 0 \qquad n \in Z \tag{3.15}$$

$$[H_n, \phi^j_{-j}(z)] = -q^{k(n-|n|/2)} \frac{[2jn]}{\sqrt{2}n} z^n \phi^j_{-j}(z) \qquad n \in Z\backslash\{0\} \tag{3.16}$$

$$K \phi^j_{-j}(z) K^{-1} = q^{-2j} \phi^j_{-j}(z) \tag{3.17}$$

$$\phi^j_{m+1}(z) = \frac{1}{[j+m+1]} [\phi^j_m(z), E^+_0]_{q^{-2m}} \tag{3.18}$$

where the quantum commutator $[A, B]_x$ is defined by

$$[A, B]_x = AB - xBA. \tag{3.19}$$

As in the case of type I vertex operators [10], the system of equations (3.15)–(3.17) has two independent solutions for $\phi^j_{-j}(z)$ in terms of the bosonic Heisenberg elements (2.4). We denote them respectively by $\phi^j_{-j}(z)$ and $\hat{\phi}^j_{-j}(z)$. They are given by

$$
\phi^j_{-j}(z) = \exp\left\{ -\mathrm{i}j\sqrt{\frac{2}{k}}\xi^1(z) - \frac{2\mathrm{i}j(k+1)}{\sqrt{k(k+2)}}\xi^2(z) + 2\mathrm{i}j\xi^3(z) \right\}
$$

$$
\hat{\phi}^j_{-j}(z) = \exp\left\{ -\mathrm{i}j\sqrt{\frac{2}{k}}\xi^1(z) + \frac{\mathrm{i}(k-2j)}{\sqrt{k(k+2)}}\hat{\xi}^2(z) \right\}
$$

(3.20)

where

$$
\begin{aligned}
\xi^1(z) &= \Omega^1(2, k; 2j|k, -\tfrac{1}{2}k, -k| - |z) \\
\xi^2(z) &= \Omega^2(k+2, k, 1; 2j, k+1|k, -\tfrac{1}{2}k, -k| - |z) \\
\hat{\xi}^2(z) &= \Omega^2(k+2, k; k-2j|k, -\tfrac{1}{2}k, -k| - |z) \\
\hat{\xi}^3(z) &= \Omega^3(2, 1; 2j|k, 0, 2| - |z).
\end{aligned}
$$

(3.21)

The components $\phi^j_m(z)$ and $\hat{\phi}^j_m(z)$ are derived from $\phi^j_{-j}(z)$ and $\hat{\phi}^j_{-j}(z)$ through (3.18). Henceforth, we will refer to $\phi^j_m(z)$ and $\hat{\phi}^j_m(z)$ as vertex operators and conjugate vertex operators respectively.

Let us now define the Fock spaces on which the above operators are acting. A left Fock module $F(n_1, n_2, n_3)$ and a right Fock module $\bar{F}(n_1, n_2, n_3)$ labelled by the integers n_1, n_2 and n_3 are defined by

$$
\begin{aligned}
F(n_1, n_2, n_3) &= F_-|n_1, n_2, n_3\rangle \\
F_+|n_1, n_2, n_3\rangle &= 0 \\
\bar{F}(n_1, n_2, n_3) &= \langle n_1, n_2, n_3|F_+ \\
\langle n_1, n_2, n_3|F_- &= 0
\end{aligned}
$$

(3.22)

where $F_\mp$ are respectively generated by $\{a^1_{\mp n}, a^2_{\mp n}, a^3_{\mp n}, , n > 0\}$. The states $|n_1, n_2, n_3\rangle$ and $\langle n_1, n_2, n_3|$ are defined by

$$
\begin{aligned}
|n_1, n_2, n_3\rangle &= \exp\left\{ \frac{n_1}{\sqrt{2k}}\mathrm{i}a^1 + \frac{n_2}{\sqrt{k(k+2)}}\mathrm{i}a^2 + n_3\mathrm{i}a^3 \right\} |0\rangle \\
\langle n_1, n_2, n_3| &= \langle 0| \exp\left\{ -\frac{n_1}{\sqrt{2k}}\mathrm{i}a^1 - \frac{n_2}{\sqrt{k(k+2)}}\mathrm{i}a^2 - n_3\mathrm{i}a^3 \right\}
\end{aligned}
$$

(3.23)

where the vacua states $|0\rangle = |0, 0, 0\rangle$ and $\langle 0| = \langle 0, 0, 0|$ are such that

$$
\begin{aligned}
a^i_n|0\rangle &= 0 & n \geqslant 0, \; i = 1, 2, 3 \\
\langle 0|a^i_n &= 0 & n \leqslant 0, \; i = 1, 2, 3.
\end{aligned}
$$

(3.24)

From the intertwining relations (3.15)–(3.17) it is clear that the following states are $U_q(\widehat{su}(2))$ lowest weight states:

$$
|-2j, -2j(k+1), 2j\rangle =: \phi^j_{-j}(0) : |0\rangle = \exp\left\{ -\frac{2j}{\sqrt{2k}}\mathrm{i}a^1 - \frac{2j(k+1)}{\sqrt{k(k+2)}}\mathrm{i}a^2 + 2j\mathrm{i}a^3 \right\} |0\rangle
$$

$$
|-2j, k-2j, 0\rangle =: \hat{\phi}^j_{-j}(0) : |0\rangle = \exp\left\{ -\frac{2j}{\sqrt{2k}}\mathrm{i}a^1 + \frac{k-2j}{\sqrt{k(k+2)}}\mathrm{i}a^2 \right\} |0\rangle.
$$

(3.25)

As usual the symbol $::$ denotes the bosonic normal ordering, which is defined so that the modes $\{a^i_n; n \geqslant 0, i = 1, 2, 3\}$ are always placed to the right of the modes

$\{a_n^i; n < 0, i = 1, 2, 3\}$. Moreover, one can easily check that $(E_0^+)^{2j}$ maps the state $|-2j, -2j(k+1), 2j\rangle$ to $|2j, 2j, 0\rangle$, which is a $U_q(\widehat{su}(2))$ highest weight state. The states $|2j, 2j, 0\rangle$ and $|-2j, k-2j, 0\rangle$ define respectively the Fock modules $F(2j, 2j, 0)$ and $F(-2j, k-2j, 0)$.

The currents $E^\pm(z)$ and the vertex operators $\phi_m^j(z)$ and $\hat{\phi}_m^j(z)$ map a Fock module $F(n_1, n_2, n_3)$ as:

$$E^\pm(z) : F(n_1, n_2, n_3) \rightarrow F(n_1 \pm 2, n_2 \pm (k+2), n_3 \mp 1) \tag{3.26}$$

$$\phi_m^j(z) : F(n_1, n_2, n_3) \rightarrow F(n_1 + 2m, n_2 + m(k+2) - jk, n_3 + j - m) \tag{3.27}$$

$$\hat{\phi}_m^j(z) : F(n_1, n_2, n_3) \rightarrow F(n_1 + 2m, n_2 + k(j+1) + m(k+2), n_3 - j - m). \tag{3.28}$$

One can show that the action of the currents is well defined (single valued) on the Fock modules $F(n_1, n_2, n_3)$ provided that the following condition is satisfied:

$$n_1 - n_2 \in kZ. \tag{3.29}$$

Relation (3.26) implies that the representations on which the currents are acting are the complete Fock spaces

$$\mathcal{F}(j) = \bigoplus_{r \in Z} F(2j + 2r, 2j + r(k+2), -r) \tag{3.30}$$

$$\hat{\mathcal{F}}(j) = \bigoplus_{r \in Z} F(-2j + 2r, k - 2j + r(k+2), -r). \tag{3.31}$$

Note, however, that the irreducible $U_q(\widehat{su}(2))$ highest (lowest) weight representation is only a subspace embedded in the Fock space $\mathcal{F}(j)(\hat{\mathcal{F}}(j))$ and can be projected out through a BRST analysis [11]. We claim that the dual representation of the highest weight Fock space $\mathcal{F}(j)$ is isomorphic as a $U_q(\widehat{su}(2))$ module to the lowest weight Fock space $\hat{\mathcal{F}}(j)$. Moreover, we make the normalization

$$\langle 2j, 2j(k+1) + k, -2j | 2j, 2j, 0\rangle = 1. \tag{3.32}$$

As far as we know, there is no algebraic proof of this claim even in the classical case, i.e. the $su(2)$ conformal field theory [5] (see however [12] for a proof in the case of the Virasoro algebra). We will nevertheless check its validity through the two-point matrix element.

The next ingredient we need is the action of a screening operator denoted by $S(z)$. This is an operator that commutes with the currents $E^\pm(z)$, $\psi(z)$ and $\phi(z)$ up to total quantum derivatives. According to [13, 14], here are three such operators in the case of $U_q(\widehat{su}(2))$. However, for our purposes here we need only one of them. It is given by

$$S(z) = {}_1\mathcal{D}_z(\exp\{-i\eta^3(z)\}) \exp\left\{i\sqrt{\frac{k}{k+2}} \eta^2(z)\right\} \tag{3.33}$$

with

$$\begin{aligned} \eta^2(z) &= \Omega^2(k+2, 1; 1| -k-2, -\tfrac{1}{2}k, 0| + |z) \\ \eta^3(z) &= \Omega^3(2, 1; 1| -k-2, 0, k+2| + |z) \end{aligned} \tag{3.34}$$

and satisfies the following commutation relations:

$$\begin{aligned} &[\varphi(z), S(w)] = 0 \\ &[\psi(z), S(w)] = 0 \\ &[E^+(z), S(w)] = 0 \\ &[E^-(z), S(w)] = -{}_{k+2}\mathcal{D}_w\left(\frac{h(w)}{z-w}\right) \end{aligned} \tag{3.35}$$

for some operator $h(w)$. The quantum derivative ${}_k\mathcal{D}_z f(z)$ of a function $f(z)$ is as usual defined by:

$$
{}_k\mathcal{D}_z f(z) = \frac{f(zq^k) - f(zq^{-k})}{z(q - q^{-1})}.
\tag{3.36}
$$

Relations (3.35) imply that the screening charge Q, which is constructed as a Jackson integral of the screening current, i.e.,

$$
Q = \int_0^{s\infty} \mathrm{d}_p z S(z)
\tag{3.37}
$$

with $p = q^{2(k+2)}$, commutes with $U_q(\widehat{su}(2))$. The Jackson integral of a function $f(z)$ is defined by [4]:

$$
\int_0^{s\infty} \mathrm{d}_p z f(z) = s(1 - p) \sum_{n \in Z} f(sp^n) p^n.
\tag{3.38}
$$

The action of the screening charges (3.37) on a Fock module $F(n_1, n_2, n_3)$ is given by

$$
Q : F(n_1, n_2, n_3) \to F(n_1, n_2 + k, n_3 - 1).
\tag{3.39}
$$

Because of the above properties of Q and the relations (3.27)–(3.28), one can construct the following two screened vertex operators, which map $\mathcal{F}(j_1)$ to $\mathcal{F}(j_2)$ and $\hat{\mathcal{F}}(j_2)$ respectively,

$$
\tilde{\Phi}_{j_1}^{j\,j_2}(z) = g_{j_1}^{j\,j_2}(z) \sum_{m=-j}^{j} Q^r \phi_m^j(z) \otimes v_m^j
\tag{3.40}
$$

$$
\hat{\tilde{\Phi}}_{j_1}^{j\,j_2}(z) = \tilde{g}_{j_1}^{j\,j_2}(z) \sum_{m=-j}^{j} Q^s \hat{\phi}_m^j(z) \otimes v_m^j
\tag{3.41}
$$

where $r = j + j_1 - j_2$, $s = j_1 + j_2 - j$, and $g_{j_1}^{j\,j_2}(z)$ and $\tilde{g}_{j_1}^{j\,j_2}(z)$ are normalization functions that can be determined from (3.14). Note that we can also construct two more screened vertex operators that map $\hat{\mathcal{F}}(j_1)$ to $\mathcal{F}(j_2)$ and $\hat{\mathcal{F}}(j_2)$, but since we do not need them in the sequel we will not consider them here.

4. N-point matrix element

The physically interesting situation is when the vertex operators carry spin $j = \frac{1}{2}$, in which case we denote them by $\Phi_{j_1}^{j_2}(z) \equiv \Phi_{j_1}^{1/2\,j_2}(z)$. They map $\mathcal{F}(j_1)$ to $\mathcal{F}(j_1 \pm \frac{1}{2})$ and $\hat{\mathcal{F}}(j_1 \pm \frac{1}{2})$. The N-point matrix element we are interested in is the vacuum-to-vacuum expectation value $\langle 0| \Phi_{j_{n-1}}^{j_n}(z_n) \Phi_{j_{n-2}}^{j_{n-1}} - 1(z_{n-1}) \dots \Phi_{j_1}^{j_2}(z_2) \Phi_{j_0}^{j_1}(z_0)|0\rangle$. Here $|0\rangle$ is identified with the highest weight vector of $V(\Lambda_0)$. This matrix element is represented by the following sum of matrix elements in the Fock spaces:

$$
\langle 0| \Phi_{j_{n-1}}^{j_n}(z_n) \Phi_{j_{n-2}}^{j_{n-1}}(z_{n-1}) \dots \Phi_{j_1}^{j_2}(z_2) \Phi_{j_0}^{j_1}(z_1)|0\rangle
$$

$$
= \sum_{\{m_1,\dots,m_n\}} \Bigg\{ \left(\prod_{i=1}^{n} z_i^{\Delta_{J_i} - \Delta_{J_{i-1}}} g_{j_{i-1}}^{j_i}(z_i) \right) \langle 0| Q^{s_n} \hat{\phi}_{m_n}^j(z_n) Q^{r_{n-1}} \phi_{m_{n-1}}^j(z_{n-1})
$$

$$
\cdots Q^{r_2} \phi_{m_2}^j(z_2) Q^{r_1} \phi_{m_1}^j(z_1)|0\rangle v_{m_n}^{j_n} \otimes \cdots \otimes v_{m_1}^{j_1} \Bigg\}
\tag{4.42}
$$

where

$$j = \tfrac{1}{2}$$
$$j_0 = j_n = 0$$
$$j_1 = j_{n-1} = \tfrac{1}{2}$$
$$j_i = j_{i-1} \pm \tfrac{1}{2} \qquad i = 1, \ldots, n$$
$$r_1 = s_n = 0$$
$$r_i = j_{i-1} - j_i + \tfrac{1}{2} \qquad i = 1, \ldots, n-1.$$

(4.43)

The normalization constants $g^{j_i}_{j_{i-1}}(z) \equiv g^{1/2 \, j_2}_{j_1}(z)$ for $i = 1, \ldots, n-1$ have been calculated in [6] and $g^{j_n}_{j_{n-1}}(z) = \tilde{g}^0_{1/2}(z) \equiv \tilde{g}^{1/2 \, j_2}_{j_1}(z)$ is computed from the requirement that

$$\tilde{g}^0_{1/2}(z)\hat{\phi}^{1/2}_{-1/2}(z)|\Lambda_1\rangle = |\Lambda_0\rangle + \cdots.$$

(4.44)

Here the state $|\Lambda_0\rangle = |0, k, 0\rangle$ is the highest weight of $V(\Lambda_0)$ which is embedded in $\hat{\mathcal{F}}(0)$. We find

$$\tilde{g}^0_{1/2}(z) = (-zq^k)^{3/[2(k+2)]}.$$

(4.45)

Note that the consistency between the relations (4.42) and (3.32) imposes the following two constraints:

$$\sum_{i=1}^{n} m_i = 0 \qquad L \equiv \sum_{i=1}^{n-1} r_i + s_n = j(n-2) = \frac{n-2}{2}$$

(4.46)

where L denotes the total number of screening charges. This means that the only non-vanishing vacuum-to-vacuum matrix elements are those of an even number of vertex operators. Moreover, should one use a vertex operator instead of a conjugate vertex operator in (4.42), and the same bosonic vacuum for both a $U_q(\widehat{su}(2))$ representation and its dual, i.e., $\langle 0|0\rangle = 1$, then the above constraints would have become

$$\sum_{i=1}^{n} m_i = 0 \qquad L \equiv \sum_{i=1}^{n} r_i = jn = \tfrac{1}{2}n.$$

(4.47)

Therefore by using the conjugate vertex operator we have removed one redundant integral. Since all the operators inside this expectation value are realized explicitly in terms of bosonic modes, one can easily normal order them by computing all the operator product expansions (OPEs). The c-function that we get from these OPEs is precisely the expectation value we are interested in (the vacuum-to-vacuum matrix element of the normal ordered exponential operators is equal to 1). One should note, however, that the resulting integral expression of this expectation value is not always well defined because the integrands are not necessarily convergent due to the singularities that arise in the OPEs and clearly a kind of regularization to remove any divergence is needed. This issue has been addressed partially in [6] but further analysis is required.

A particular case where the above problem does not arise simply because the insertion of screening charges is not required (indeed, if $n = 2$ then relation (4.46) implies that $L = 0$), and where the computation can be made more explicit, is provided by the two-point matrix element $\langle 0|\Phi^0_{1/2}(z)\Phi^{1/2}_0(w)|0\rangle$. This expectation value is represented as a vacuum expectation value over the Fock spaces by

$$\langle 0|\Phi^0_{1/2}(z)\Phi^{1/2}_0(w)|0\rangle = z^{\Delta_0-\Delta_{1/2}} w^{\Delta_{1/2}-\Delta_0} \tilde{g}^0_{1/2}(z)g^{1/2}_0(w)$$
$$\times (\langle 0|\hat{\phi}_+(z)\phi_-(w)|0\rangle v_+ \otimes v_- + \langle 0|\hat{\phi}_-(z)\phi_+(w)|0\rangle v_- \otimes v_+)$$

(4.48)

with

$$\phi_\pm(z) \equiv \phi_{\pm 1/2}^{1/2}(z) \qquad \hat{\phi}_\pm(z) \equiv \hat{\phi}_{\pm 1/2}^{1/2}(z)$$

$$v_\pm \equiv v_{\pm 1/2}^{1/2} \qquad \Delta_0 = 0 \qquad \Delta_{1/2} = \frac{3}{4(k+2)}. \tag{4.49}$$

$\tilde{g}_{1/2}^0(z)$ is given by (4.45) whereas $g_0^{1/2}(z)$ can be obtained from the normalization

$$g_0^{1/2}(z)\phi_+(z)|\Lambda_0\rangle = |\Lambda_1\rangle + \cdots. \tag{4.50}$$

Using the explicit form of $\phi_+(z)$, which is given below we find

$$g_0^{1/2}(z) = -q^{-1}. \tag{4.51}$$

The explicit bosonic constructions of the vertex operators $\phi_\pm(z)$ and $\hat{\phi}_\pm(z)$ that we obtain by solving the intertwining conditions (3.12) are given by

$$\phi_-(z) = \exp\left(-\frac{i}{\sqrt{2k}}\xi^1(z) - \frac{i(k+1)}{\sqrt{k(k+2)}}\xi^2(z) + i\xi^3(z)\right)$$

$$\phi_+(z) = [\phi_-(z), E_0^+]_q$$

$$= q^{k+2}\oint_{|w|<|zq^{1+k}|}\frac{dw}{2\pi i}\frac{z : \phi_-(z)E_+^+(w) :}{w(w-zq^{1+k})} - q^k\oint_{|w|>|zq^{k-1}|}\frac{dw}{2\pi i}\frac{z : \phi_-(z)E_-^+(w) :}{w(w-zq^{k-1})} \tag{4.52}$$

$$\hat{\phi}_-(z) = \exp\left(-\frac{i}{\sqrt{2k}}\xi^1(z) + \frac{i(k-1)}{\sqrt{k(k+2)}}\hat{\xi}^2(z)\right)$$

$$\hat{\phi}_+(z) = [\hat{\phi}_-(z), E_0^+]_q$$

$$= q\oint_{|w|<|zq^{1+k}|}\frac{dw}{2\pi i}\frac{: \hat{\phi}_-(z)E_-^+(w) :}{w-zq^{1+k}} - q\oint_{|w|>|zq^{k-1}|}\frac{dw}{2\pi i}\frac{: \hat{\phi}_-(z)E^+(w) :}{w-zq^{k-1}}.$$

In deriving relations (4.52) we have used the following OPEs:

$$E_+^+(z)\phi_-(w) = q : E_+^+(z)\phi_-(w) :$$

$$\phi_-(z)E_+^+(w) = \frac{z-wq^{1-k}}{z-wq^{-k-1}} : \phi_-(z)E_+^+(w) : \qquad |wq^{-k-1}| < |z|$$

$$E_+^+(z)\hat{\phi}_-(w) = \frac{1}{z-wq^{k-1}} : E_+^+(z)\hat{\phi}_-(w) : \qquad |wq^{k-1}| < |z|$$

$$\hat{\phi}_-(z)E_+^+(w) = -\frac{q^{-k}}{z-wq^{-k-1}} : E_+^+(w)\hat{\phi}_-(z) : \qquad |wq^{-k-1}| < |z| \tag{4.53}$$

$$E_-^+(z)\phi_-(w) = q^{-1}\frac{z-wq^{k+1}}{z-wq^{k-1}} : E_-^+(z)\phi_-(w) : \qquad |wq^{k-1}| < |z|$$

$$\phi_-(z)E_-^+(w) =: \phi_-(z)E_-^+(w) :$$

$$E_-^+(z)\hat{\phi}_-(w) = \frac{1}{z-wq^{k-1}} : E_-^+(z)\hat{\phi}_-(w) : \qquad |wq^{k-1}| < |z|$$

$$\hat{\phi}_-(z)E_-^+(w) = -\frac{q^{-k}}{w-zq^{-k-1}} : E_-^+(w)\phi_-(z) : \qquad |wq^{-k-1}| < |z|.$$

Note that when $k = 1$ the conjugate vertex operator $\hat{\phi}_-(z)$ satisfies exactly the same OPEs with the currents $E^\pm(z)$ as those of the vertex operator which is bosonized through the Frenkel–Jing realization of $U_q(\widehat{su}(2))$ [15,1]. In this sense, unlike $\phi_\pm(z)$, $\hat{\phi}_\pm(z)$ are the natural generalizations of the vertex operators found in the case $k = 1$ in [15]. In order to

compute the above two-point matrix element we need to use, besides the OPEs (4.53), the following OPE:

$$\hat{\phi}_-(z)\phi_-(w) = -q^k(-zq^k)^{-3/[2(k+2)]}(z - wq^{-2})$$
$$\times \prod_{n=0}^{\infty} \frac{(p^n z/w)_\infty (p^n q^4 z/w)_\infty}{(p^n q^{-2} z/w)_\infty (p^n q^6 z/w)_\infty} : \hat{\phi}_-(z)\phi_-(w) : \qquad (4.54)$$

where $p = q^{2(k+2)}$ and

$$(a)_\infty = \prod_{n=0}^{\infty}(1 - aq^{4n}). \qquad (4.55)$$

Using all the OPEs given by (4.53) and (4.54), and carrying out the integrals we find that the above two-point matrix element takes the following simple form:

$$\langle 0|\Phi_{1/2}^0(z)\Phi_0^{1/2}(w)|0\rangle = f(w/z)(v_- \otimes v_+ - q^{-1}v_+ \otimes v_-) \qquad (4.56)$$

where

$$f(z) = z^{3/[4(k+2)]} \prod_{n=0}^{\infty} \frac{(p^n z)_\infty (p^n q^4 z)_\infty}{(p^n q^{-2} z)_\infty (p^n q^6 z)_\infty}. \qquad (4.57)$$

It can be easily checked that the two-point matrix element $\langle 0|\Phi_{1/2}^0(z)\Phi_0^{1/2}(w)|0\rangle$ as given in (4.56) satisfies the following q-KZ equations [2–4]:

$$\langle 0|\Phi_{1/2}^0(z)\Phi_0^{1/2}(pw)|0\rangle = (1 \otimes K^{-1})R^+(w/z)\langle 0|\Phi_{1/2}^0(z)\Phi_0^{1/2}(w)|0\rangle$$
$$\langle 0|\Phi_{1/2}^0(pz)\Phi_0^{1/2}(pw)|0\rangle = (K^{-1} \otimes K^{-1})\langle 0|\Phi_{1/2}^0(z)\Phi_0^{1/2}(w)|0\rangle \qquad (4.58)$$

where the R matrix $R^+(z)$ is given by

$$R^+(z) = \rho(z)\bar{R}(z) \qquad (4.59)$$

with

$$\rho(z) \qquad = q^{-1/2}\frac{(q^2 z)_\infty^2}{(z)_\infty (q^4 z)_\infty}$$
$$\bar{R}(z)v_\pm \otimes v_\pm = v_\pm \otimes v_\pm$$
$$\bar{R}(z)v_+ \otimes v_- = \frac{(1 - z)q}{1 - q^2 z}v_+ \otimes v_- + \frac{(1 - q^2)z}{1 - q^2 z}v_- \otimes v_+ \qquad (4.60)$$
$$\bar{R}(z)v_- \otimes v_+ = \frac{1 - q^2}{1 - q^2 z}v_+ \otimes v_- + \frac{(1 - z)q}{1 - q^2 z}v_- \otimes v_+.$$

Moreover, when we set $k = 1$, this two-point matrix element reduces to

$$\langle 0|\Phi_{1/2}^0(z)\Phi_0^{1/2}(w)|0\rangle = (w/z)^{1/4}\frac{(w/z)_\infty}{(q^{-2}w/z)_\infty}(v_- \otimes v_+ - q^{-1}v_+ \otimes v_-) \qquad (4.61)$$

which is just the result found in [2] up to a similarity transformation of the $R^+(z)$ matrix by the transposition matrix P, which is defined by

$$P(v_\pm \otimes v_\pm) = v_\pm \otimes v_\pm \qquad P(v_\pm \otimes v_\mp) = v_\mp \otimes v_\pm. \qquad (4.62)$$

5. Conclusions

In this paper we derive two independent bosonizations of each dynamical vertex operator of the higher spin XXZ model. Both of them satisfy the same intertwining relations. We refer to one of them as the vertex operator and the other as its conjugate vertex operator. This terminology is due to our claim that when the former acts on the left and the latter acts on the right bosonic vacua, they create respectively a $U_q(\widehat{su}(2))$ left representation and its dual. We confirm the validity of this claim through the two-point matrix elements. As a result, the N-point matrix elements of dynamical vertex operators do not involve redundant Jackson integrals coming from the insertion of screening charges. It would be interesting to find an algebraic proof to the above claim. Finally, it is natural to expect that the use of conjugate vertex operators would also simplify the computation of other important physical quantities like form factors (traces of products of type I and type II vertex operators) and correlation functions (traces of type I vertex operators) of local operators of the higher spin XXZ model. This is presently under investigation.

Acknowledgments

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